Art. #2443, 10 pages, https://doi.org/10.15700/saje.v45ns2a2443

A commognitive analysis of pre-service mathematics teachers' definitions and classification of quadrilaterals

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Abstract
Definition of geometric concepts forms a critical part of teachers' content knowledge for the teaching of geometry. The purpose of the study reported on here was to analyse pre-service mathematics teachers' geometric thinking with regards to definitions and classification of quadrilaterals. Data were generated from 8 pre-service teachers based on their performance in a test administered to those who voluntarily accepted to take part in the study. The 8 participants were divided into 2 groups of 4 each – 4 in Group A who performed well in the test, and 4 in Group B whose performance was not so good. The results show that even though the pre-service teachers (PSTs) had difficulties with definitions, those in Group A demonstrated moderate thinking on the use of literate words in definitions of quadrilaterals compared to their counterparts in Group B who showed deficiencies and used colloquial words in their definitions. We also found that a few of the participants in Group A showed an explorative way of thinking in the classification of quadrilaterals. Thus, generally, the PSTs' geometric thinking on inclusion relations of quadrilaterals was more ritualised in nature.

Keywords: classifications; commognitive; definition; pre-service teachers; quadrilaterals

Introduction
Geometry, an indispensable topic in mathematics, is noted for empowering students by fostering their reasoning skills essential for developing their mathematical thinking (Bonyah & Larbi, 2021; Jablonski & Ludwig, 2023). The National Council of Teachers of Mathematics ([NCTM], 2000:41) expects that learners will be able to "analyse characteristics and properties of two-and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships." Also, learners should be able to identify geometric shapes and describe their attributes using appropriate vocabulary, define geometric shapes, develop spatial thinking, use their visual ability and develop deductive thinking, among other skills (Jablonski & Ludwig, 2023; Jones, 2000; Sinclair, Cirillo & De Villiers, 2017). Learners' development of geometric habits of mind, such as spatial reasoning and visualisation, are critical tools for learning mathematics (Yurmalia & Herman, 2021). Jones (2000) maintains that much of the learner's thinking necessary for learning mathematics is spatial in nature. Research shows a significant relationship between geometric understanding and mathematical competencies (Robichaux-Davis & Guarino, 2016). This suggests that improved mathematics performance can be realised if special attention is given to the teaching and learning of geometry.

Due to the importance of geometry in our daily lives, it is included in the mathematics curricula of many countries (Bonyah & Larbi, 2021; Jones, 2000; Luneta, 2015; Serin, 2018). However, a growing body of research shows that many learners find geometry the most difficult and confusing aspect of the mathematics curriculum (Luneta, 2015; Naidoo & Kapofu, 2020; Ngirishi & Bansilal, 2019).

In Ghana, like many other countries, many learners find the subject matter of geometry challenging to learn. In the Chief Examiner's diagnostic analysis of learners' written scripts to determine their strengths and weaknesses in the various content areas of mathematics, which is reported by the West African Examinations Council (WAEC), geometry has, for many years, been among the areas that have been repeatedly highlighted as learners' major weakness (WAEC, 2015, 2017, 2018). In 2021, a similar weakness was found among the candidates' solutions of problems on mensuration and geometry (WAEC, 2021). The reports add that most candidates do not attempt questions on geometry when given the choice. The few who do attempt to answer these questions often display errors that highlight their difficulty in understanding the subject matter. To improve learners' understanding in learning geometry and to make them appreciate it, teachers need to possess deep and flexible content knowledge of the subject matter to enhance their teaching and learning activities in the classroom (Robichaux-Davis & Gaurino, 2016). In this study, therefore, we focused on PSTs' geometric thinking with regard to definitions and classification of quadrilaterals. According to Ndlovu (2014), developing geometric thinking starts with the definitions of geometric shapes. In this study, a quadrilateral is defined as a polygon with four sides. We also consider definitions and classification of quadrilaterals such as a parallelogram as a quadrilateral with opposite sides being parallel, and a rectangle as a parallelogram with each of its angles equal to 90°, a rhombus as a parallelogram with all sides equal in length, and a square as a rectangle with all sides equal (Avcu, 2023; Baktemur, Ayan Civak & Isiksal Bostan, 2021; Jacobs, 2020; Žilková, 2014).

The research questions that we sought to answer in this study were:

1) What are pre-service teachers' discursive geometric thinking on definitions of quadrilaterals?
2) What are pre-service teachers' routine thinking of inclusion relations of quadrilaterals?

Literature Review

Definitions of geometric concepts

Definitions play a critical role in the Euclidean axiomatic system, and they are a starting point to think geometrically (Ndlovu, 2014). The author further claims that "definitions of terms form the basis from which properties of the terms are logically defined and the means by which the user can name and classify geometric objects" (Ndlovu, 2014:6642). This could mean that definitions form the starting point of learning geometry, which makes mathematics teachers' knowledge and competencies in this area critical for effective teaching.

Research, however, shows that PSTs find it difficult to complete tasks on mathematical definitions, especially in geometry. They are unable to construct statements that portray the exact description of geometric or mathematical objects (Kemp & Vidakovic, 2021). Ndlovu (2014) asserts that definition of 2D shapes and their properties forms part of PSTs' difficulties in learning geometry. Fujita and Jones (2006) found that PSTs' major difficulty was their inability to provide definitions for the types of quadrilaterals, even though they were able to correctly draw those shapes. When a little more than half (58.9%) of the participants correctly defined a parallelogram, the authors attributed it to a possible reminder of "parallel lines" by the name "parallelogram." Those who provided correct definitions for the other types of quadrilaterals (squares, rectangles and trapeziums), were 38%, 21.5% and 12%, respectively.

Pre-service teachers' understanding of quadrilaterals

Several authors report that PSTs' have difficulty defining and understanding the inclusion relations of classifying quadrilaterals (Avcu, 2023; Baktemur et al., 2021; Ndlovu, 2014; Rianasari, Julie & Sulistyani, 2016; Ulger & Broutin, 2017; Žilková, 2014). Quadrilaterals is one of the topics taught in secondary school mathematics, so it is expected that teachers and PSTs understand the definitions and inclusion relations among them (Rianasari et al., 2016). However, studies show that many PSTs have difficulties defining an inclusive classification of quadrilaterals (Baktemur et al., 2021; Ndlovu, 2014; Ulger & Broutin, 2017; Žilková, 2014).

For example, in a study on pre-service mathematics teachers' understanding of quadrilaterals and their relationships, Ulger and Broutin (2017) posed open-ended questions to 27 participants through clinical interviews. The analysis shows that there were more personal definitions (colloquial use of words) in their responses, than formal definitions (literate use of words). Also, the PSTs defined the shapes based on their experiences which showed more of the so-called "prototype figures." According to Fujita (2012), prototype figures are those formed during the learners' first (and subsequent) encounter with such geometric objects. Ulger and Broutin (2017) claim that many participants showed difficulties with inclusion relations within quadrilaterals, of which literature shows similar findings. The authors reported that most PSTs used incorrect properties in their definitions. For example, some said: "a parallelogram does not have right angles" and "a parallelogram is an oblique quadrilateral." The authors explain that those definitions were based on prototype figures.

Baktemur et al.'s (2021) investigation into PSTs' conceptions and misconceptions of definitions, classifications, and inclusions involved 20 purposely selected participants who were to take a course in the methodology of teaching mathematics after they had completed various content modules. Two weeks of instruction, devoted to teaching geometry, focused on Van Hiele's geometric thinking on two- and three-dimensional shapes and related properties. The PSTs were asked to design tasks that would support learning and understanding. Analysis of data generated before and after the module revealed that PSTs showed inadequate understanding of hierarchical classification of quadrilaterals and experienced difficulties with definitions. The PSTs also described quadrilaterals with no indication of inclusion relations among the shapes. The authors indicate that after the methodology course there was much improvement in the PSTs' definition and classification.

Within Sfard's (2008) commognitive theory, learners may think that geometric shapes may not be referred to by more than one name. They think that "a square cannot be called a rectangle or a parallelogram." This is a demonstration of learners' ritualised way of thinking which often happens due to strict rules of definition (Sfard, 2008), or an encounter with geometric objects in previous learning (Fujita, 2012; Ulger & Broutin, 2017).

Sixteen pre-service mathematics teachers participated in another study by Ndlovu (2014). Data about the participants' definitions and inclusion relations of quadrilaterals revealed that, although they showed some understanding of definitions, their thinking on inclusion relations was limited. Ndlovu (2014) concludes that the participants' understanding of quadrilaterals was poorer than expected. Most of the PSTs thought that a rectangle was not a parallelogram, and a square was not a rhombus (Ndlovu, 2014). According to Rianasari et al. (2016), PSTs often prefer partition classification to hierarchical classification because in partition they do not consider shapes that share properties with others.

Theoretical Framework

Commognition stems from the theory that a person engages in self-communication by thinking when performing an activity (Sfard, 2008). According to Sfard (2008), thinking is an individualised form of communication. Thus, one communicates with oneself while thinking. Communication, on the other hand, is "a collectively performed, rule-driven activity" that directs the activities of people or oneself (Sfard, 2008:118). The author stresses that the processes of communication and individual cognitive processes (thinking) are different occurrences of the same phenomenon. The commognitive theory is a term coined to show the interrelationship between thinking processes and communication. The theory provides a discursive framework for studying and interpreting an individual's activity to gain insight into "intricacies of learning" (Sfard, 2008:288).

Sfard (2008, 2015) identifies the elements of mathematical discourse as: word use, visual mediators, routines, and narratives. She explains that a discourse is characterised by the kinds of keywords used. It refers to the vocabularies or terminologies of communication. Communication is a major tool through which teachers convey knowledge and ideas to learners. The kind of words used in a discourse gives it a unique feature. Sfard (2008, 2015) classifies word use as either literate or colloquial. She explains that words, terms, or vocabularies used in mathematical discourse that have specific mathematical meaning are called "literate", while colloquial discourse emerges spontaneously from daily conversation and may have different meanings to different people.

Visual mediators are pictorial materials used to communicate geometric and mathematical concepts. They provide participants with a visual tool to talk about when constructing geometric ideas. Visual mediators, according to Sfard (2008:147), are imagery materials that help "discussants to identify the object of their talk" and steer communication among them. Examples of visual mediators are variables (algebraic symbols), graphs, sketches (drawings), diagrams, numbers, formulae, manipulatives, et cetera. Diagrams play a significant role in geometric discourse. According to Samkoff, Lai and Weber (2012), diagrams are visual forms used to present information in mathematics. Watson and Harel (2013) and Yahya, Kurniawan, Shamsuddin and Zain (2022) claim that geometry is one of the content areas in mathematics in which diagrams are mostly used to convey concepts.

Routines may be described as following a procedure or distinctive pattern in a geometric discourse. They are repetitive patterns (Berger, 2013) that are characterised by a given discourse. Sfard (2008:208) defines routines as "a set of meta-rules that describe a repetitive discursive action" and can be ritualised or explorative. Sfard (2008) asserts that rituals are the distinctive rules that guide an action and are pre-determined by people in authority. Rituals are the basic units of a discourse and deal with "how" (step-by-step guide) to get something done. Explorative discourse is the learner's ability to explain "when" to use a routine and "why" the routine works. Exploration is the implicit or explicit understanding of geometric objects of study. Learners' development of explorative routines enables them to apply knowledge in several situations.

Narratives, according to Sfard (2008:134), are "any sequence of utterances framed as a description of objects that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures." Examples are theorems, definitions, axioms, properties, theories, et cetera.

Methodology

This qualitative case study was conducted within the interpretive paradigm. A qualitative case study is often used to explore several characteristics of cases, which can be individuals or groups (Creswell, 2014). A qualitative case study design was used to gain an in-depth understanding of how PSTs have experienced geometry from pre-school to their current level of education.

Participants

The study participants were eight second-year mathematics education students studying at a university in Ghana. This year group was preferred because they had recently taken geometry content courses and method of the teaching of mathematics. The participants were among those who voluntarily accepted to take part in the study after the purpose and the processes for collecting data were explained to them. The participants were selected and grouped based on their performance in a test consisting of 12 items constructed at the senior high school level of difficulty. The criterion for good performance was set as solving two-thirds or more of the 12 questions correctly. Hence, those who solved eight or more questions were placed in Group A, while those who solved fewer than eight questions were placed in Group B. Four from each group were invited to participate in the actual study. In other words, those in Group A were considered to have performed well, and the Group B participants were those who performed not as well.

Instruments

Because of its flexibility to include follow-up questions, semi-structured interviews were used as the instrument to generate data (DeJonckheere & Vaughn, 2019). The instrument was developed based on the geometry content (quadrilaterals) of the Ghanaian mathematics curriculum for secondary schools to ensure that the items on the instrument served its purpose. The interview items were constructed based on the review of the related literature on areas required to understand the PSTs' proficient thinking of defining and classifying quadrilaterals. For example, Is a rectangle a parallelogram or not? Can you explain why you said that? Also, two secondary mathematics teachers were asked to assess the items based on the description of the mathematics curriculum. The participants were asked to define the various types of quadrilaterals and indicate their inclusion with others during the interviews. The interviews helped us to access the participants' thinking with regard to the study phenomenon (Best & Khan, 2006; Wahyuni, 2012). It also aided in understanding the words used to describe the geometric shapes and their relationships.

Trustworthiness of the Data

To enable readers to establish the authenticity or the quality of the study findings (Connelly, 2016), the following criteria were used to ensure trustworthiness of the data.

Credibility is ensuring that the participants agree with the study findings. This was done by allowing the research participants to review and verify the transcribed data as recommended by Yin (2014). The rationale was to ensure that we interpreted the participants' responses correctly. Peer debriefing was also employed to ensure credibility.

Dependability of the data was ensured by using audit trials to offer rich descriptions of the study process (Anney, 2014). This was done through frequent engagement with the participants to assess the study findings and interpretations. In addition, a detailed description of the methods employed was provided.

Transferability, which concerns with the applicability of the findings in other contexts, was ensured by providing a comprehensive and detailed description of the study. All these measures helped to ensure the confirmability of the data. These measures helped to show that the data provided by the participants were accurate and without any researcher biases.

Data Analysis

The responses provided by each PST served as the unit of analysis. The audio-recorded responses obtained were transcribed to text and analysed by labelling and organising definitions according to the characteristics of the framework in line with their use of words. As guided by the framework, the attention was on whether the participants used words in their discourse literately or colloquially. Sfard (2008) states that a word used in mathematical discourse with a shared meaning among participants is termed "literate." A vivid definition of a rectangle as a parallelogram of which the angles are 90° is considered a literate definition, while defining a rectangle as a quadrilateral with equal opposite sides is considered as colloquial – as seen in the PSTs' responses.

Ethical Considerations

Permission to conduct this study was sought from the head of the department. After the purpose of the study and the procedures for data generation were explained to the PSTs, they were invited to participate. They were assured of confidentiality of any data obtained as well as anonymity, since pseudonyms were to be used in the report. Thus, participation in the study was voluntary.

Findings and Discussion

Use of Words in Definitions of Quadrilaterals

The use of words plays a significant role in mathematical discourse. It provides an avenue to describe specific ideas in a discourse, especially definitions of geometric terms, concepts and theorems. Sfard (2008) claims that mathematical words have a shared or a specialised meaning among participants within a discourse, in which case they are used literately, while nonspecialised words are used colloquially. The analysis shows that all the PSTs shared their understanding of quadrilaterals by indicating that a quadrilateral is a four-sided or a plane figure bound by four sides. The participants all used literate words in their definitions. The participants gave an exhaustive list of types of quadrilaterals, which they were then asked to define.

All the participants defined a parallelogram using endorsed words. They concisely defined the term "parallelogram", with either the necessary and sufficient conditions (Žilková, 2014) or with inferred properties. Of the eight participants, Clement, Jones and Stephen, all in Group A, and Cynthia, from Group B, defined a parallelogram as a four-sided figure which has two pairs of parallel sides. The definitions are shown in the following excerpts:

> *... it is a quadrilateral which has two pairs of sides parallel* (Jones, Group A).
>
> *... it is a four-sided plane figure formed from two pairs of parallel lines* (Clement, Group A).

The other participants defined it as follows:

> *... a parallelogram is a four-sided figure which has two sides and two opposite angles equal* (Maxwell, Group A).
>
> *... a parallelogram is a type of quadrilateral whose opposite sides are equal* (Albert, Group B).

The preceding definitions show that many of the PSTs in Group A defined a parallelogram with the concept of parallelism of the two sides, while those in Group B defined it using inferred properties of shapes. The definitions of the participants in Group A showed their abilities to define a parallelogram with the necessary and sufficient conditions.

Varied thinking, and hence varied use of words emerged when the participants were asked to define the types of quadrilaterals. Some of the participants used mathematically literate words in their definitions, while others used colloquial words. Two PSTs from Group A (Stephen and Clement) and Cynthia (B) provided a literate definition of a rectangle and a square. The remaining participants provided incomplete definitions, or the definitions did not show the exact description of the figure under discussion. For example, Jones (A) defined a rectangle as "*... a quadrilateral ... with opposite sides parallel*." He also defined a square as "*a quadrilateral having all sides equal*." It needs to be noted that the definition of a rectangle matches that of a parallelogram, and the definition of a square matches that of a rhombus, hence the need to specify that their angles are equal or at right angles. The other five participants defined these figures in a similar way. This finding supports that of Avcu (2023) and Fujita and Jones (2006) who found that most of the PSTs could not define these two shapes correctly. They could not exercise any logical reasoning to exclude the shape from a more general one.

In defining the types of quadrilaterals, only Stephen (A) defined a rectangle from the perspective of inclusion in a parallelogram being a four-sided figure with opposite, parallel sides. He said that "*a rectangle is a parallelogram with its interior angles being 90 degrees*." He may have considered that some concepts or properties of a rectangle also apply to a parallelogram (Rianasari et al., 2016) which informed his definition along the path of class inclusion.

Not all the PSTs were able to define a rhombus with an acceptable use of words. Stephen (A) said that a "*rhombus is a parallelogram with all sides being equal*." Cynthia (B) provided a similar understanding when she said that "*a rhombus too is a quadrilateral with all equal sides*." A deep reflection on their responses shows that Stephen (A) and Cynthia (B) demonstrated some knowledge of inclusion criteria in their definitions. Cynthia excluded a rhombus from her definition of a square when she defined it as "*a quadrilateral with all equal sides*" but carefully defined a square as "*a square has all the sides and angles equal*." She may have linked this definition to her previous definition of a rhombus.

The other participants could not define a rhombus with concise words. Jones (A) defined a rhombus with no restriction on membership. He said that "*a rhombus is a quadrilateral which has two pairs of sides being parallel*." A careful consideration of Jones' definition is more suitable for a parallelogram, for which a rhombus shares this property in its definition. Hence, it should have come with the extra word-phrase "with all sides equal" to make it unique.

The results show that the PSTs struggled to use appropriate words to define some quadrilaterals such as a rhombus or a trapezium. In addition, we found that, except for Stephen (A), almost all the participants' attempts to define the shapes showed no effort to consider the inclusion relations among the quadrilaterals. This finding supports that of Baktemur et al. (2021), who also found that the participating teachers described the types of quadrilaterals without any attention to the inclusion criteria among the shapes.

Maxwell's (A) use of words in defining several types of quadrilaterals could not be fully analysed because, at some point in time, he said that he had forgotten the definitions of some of the shapes that were central to our discussion. Maxwell was able to express some properties of a parallelogram using appropriate words but could not transfer such thinking to other related types. For example, Maxwell previously described a parallelogram as having opposite sides of equal length and the diagonals bisecting each other. When asked to define a rhombus, he responded, "*I know the figure but I have forgotten the definition. I know it is like a square. All the sides are equal, but I have forgotten the definition.*" This suggests that even though he may have seen the shape in his mind, he may not have fully internalised the characteristic features thereof, in which case, he was unable to use appropriate words to describe it in an object-driven way.

Visual Mediator

Visual mediators are graphs, tables, diagrams, symbols, or prompts that communicate certain important features of geometric shapes in a discourse (Sfard, 2008), and help in communication and reasoning. Drawing in geometric education serves as a medium to understand learners' representations of geometric concepts (Thom & McGarvey, 2015). Some of the PTSs were found to resort to drawing when they were asked to define the geometric shapes. They drew diagrams of the concepts being explored before trying to define them. This shows that diagrams play a significant role in the participants' thinking processes. In this way, the participants seemed to have attached their definitions to a visual representation of a quadrilateral, rather than the literate description (Sfard, 2008). For example, when Stephen (A) was asked to define a rectangle, he drew it, and indicated the parallelism with an iconic mediator, which he described in relation to a parallelogram.

This result shows that diagrams served as visual representations of what the participant was thinking about. The diagram probably enabled him to observe the important information he needed to express his definition in words. In this regard, the diagram could be considered a powerful tool for thinking. This supports the finding of Rizwan, Naseem and Raza (2018) that those who learned through diagrams explained their geometric ideas in a more precise way. It also supports the assertion of Brizuela and Gravel (2013) that visual representation (diagrams) is a means to capture and make meaning as well as to interpret an idea or a phenomenon. Visual representation provides an opportunity to support one's thinking and ability to process information (Lowrie, 2020). This was evident in Stephen's action of drawing before explaining, which shows that his thinking was confirmed by the diagram drawn.

Routine Thinking in the Inclusion Relations of Quadrilaterals

Sfard (2008) asserts that when an object has a set of endorsed narratives, which are subsets of another object, the objects are said to be highly similar and can be called by the same name. This can be applied to the inclusion relations of classifying quadrilaterals. Within the framework, the process of classifying quadrilaterals can be considered as either a ritualised or an explorative way of thinking.

In ritualised discourse, a shape, such as a square, is a distinct object and cannot be referred to by another name. Hence, it does not have any relationship with other shapes. However, in explorative discourse, a shape can have several names, such as a square also being called a rectangle or a parallelogram when they have similar endorsed narratives (Sfard, 2008). Defining and classifying quadrilaterals promotes learners' development of geometric thinking and mathematical argumentation (Tuset, 2019). The following sub-sections describe the PSTs' thinking about inclusion relations of quadrilaterals.

Relation Between a Rectangle and a Parallelogram

Of the eight participants, only two in Group A were able to identify that a rectangle was a parallelogram using the necessary conditions, while one other, also in Group A, used reasoning about the properties to define it. Their justification was based on the necessary condition of defining a parallelogram as a figure that has two pairs of parallel sides, or a quadrilateral with two opposite sides being parallel (Rianasari et al., 2016; Ulger & Broutin, 2017). For example, Jones (A) said, "*... because its two pairs of opposite sides are parallel*." Clement (A) also said that "*it* [rectangle] *has two pairs of opposite sides parallel*." Stephen (A), on the other hand, showed his understanding of the inclusion of a rectangle in a parallelogram by comparing some properties of the two geometric figures to justify the inclusion of a rectangle in a parallelogram:

> *A rectangle ... has two pairs of opposite and equal sides. ... a parallelogram ... a figure with two pairs of opposite and equal sides. ... has opposite interior angles of equal measure. ... looking at a rectangle, all interior angles are 90º, meaning the opposite angles are also of equal measure.* (Stephan, A)

According to Rianasari et al. (2016), showing relationships among quadrilaterals is built on exercising one's reasoning as is shown in Stephen's response. These responses from the three Group A participants indicate that they had a good grasp on the idea that rectangles were parallelograms.

Cynthia (B), on the other hand, said that a rectangle was a parallelogram, but justified it from a quadrilateral point of view. Her justification was that "*... because it is also a four-sided figure*." This incorrect justification indicates that Cynthia (B) did not understand or found it difficult to engage in relational thinking about the inclusion of rectangles in parallelograms.

In addition, Maxwell (A) and the other participants in Group B responded that a rectangle was not a parallelogram. Maxwell said that "*for the properties of a parallelogram, two opposite sides and angles are equal, but for a rectangle, I know that their opposite sides are the same, but the angles are 90 degrees, so a rectangle cannot be a parallelogram*." His response shows that he could not reason that all the angles being equal was the same or similar to opposite angles being equal. Thus, he considered the properties as isolated facts, which is classified as ritual thinking within the commognitive framework (Sfard, 2008). Alex (B) argued that a rectangle was not a parallelogram and explained that "*a rectangle has two of its opposite sides equal and parallel and a parallelogram has two of its opposite angles equal*." It should be noted that even the comparison was incorrect as the equality and parallelism of the sides of the rectangle were compared with the opposite angles in the parallelogram. Through further prompts to draw his attention to the comparison, he responded, "*even the way they look is not equal*." This could mean that his thinking about the inclusion relations of the shapes was based on their appearance and not their governing properties. This finding supports that of other researchers such as Avcu (2023), Baktemur et al. (2021) and Ngirishi and Bansilal (2019), who found that learners struggled with the class inclusion of rectangles in a parallelogram. For example, many participants in a study by Baktemur et al. (2021) did not consider a rectangle as a parallelogram at the beginning of an intervention. Also, a similar finding was reported by Ndlovu (2014), when most of the participating PSTs held similar thoughts about a rectangle and a parallelogram.

Relationship Between a Square and a Parallelogram

On the inclusion of squares in parallelograms we observed that the PSTs found it difficult to accept the inclusion of this shape. Of the eight participants, only three in Group A accepted the inclusion of squares in a parallelogram. Jones' (A) and Clement's (A) responses showed knowledge of inclusion among the two, with the justification of parallelism of opposite sides. Stephen's justification was based on a reasoning about the properties of the two shapes.

The five others either agreed or disagreed, while giving incorrect justifications. For example, Cynthia still justified her response with the concept of both shapes being quadrilaterals. She said that "*a square is a parallelogram because a square also is a four-sided figure as a parallelogram*." When questioned further to see if she could give a more acceptable reason, her response showed no clear reasoning. She stated that "*... because it* [square] *also has four sides and with the square, all the sides are equal*."

Cynthia's response shows that she may be referring to, or thinking about, partition classification instead of hierarchical classification, hence referring to the quadrilaterals. The remaining participants said that a square was not a parallelogram, giving diverse reasons such as "*... for a square, it is a type of a quadrilateral which all the sides are equal but that of a parallelogram it is not like that*" (Albert, B).

Other responses were:

> *… for parallelogram two of its sides are equal but for square all the sides are equal and the angles are also equal. But for a parallelogram only two of its opposite angles are equal* (Alex, B).
>
> *For the square all the sides are equal but for the parallelogram all the sides are not equal. Only opposite sides are equal* (Maxwell, A).

The participants' responses indicate their difficulty in accepting that a square was a parallelogram, which is a demonstration of ritual thinking. This is because of their inability to draw any relationship between the shapes (Sfard, 2008).

Relationship Between a Rhombus and a Parallelogram

Compared to previous questions, almost all the participants favourably responded to a rhombus being a parallelogram. Seven of the participants responded favourably, even though some gave incorrect justifications. Cynthia (B), as in previous responses, stated that both figures had four sides (quadrilateral), while Jones and Clement (both in A) claimed that both figures had opposite sides parallel. Stephen responded that "*a rhombus is also a special type of parallelogram. It satisfies having two opposite and equal sides and angles*."

Probing further to understand what may have guided their thinking, we observed that some of the participants drew the two shapes to support their reasoning. This could mean that their thinking may have been informed by the appearance of the two shapes, but not by the parallel nature of both sides. For example, Albert (B) said, "*both figures may have common properties because they look similar*." When Maxwell (A) drew the figures, he said, that "*... opposite angles of a rhombus are equal, and opposite angles of the parallelogram are also equal*." When questioned further, he responded by saying "*that is how I see it*."

Relationship Between a Square and a Rectangle

The participants' thinking on shape inclusion seemed to have ended with comparing the shapes to parallelograms. When we moved to comparison within shapes, particularly squares and rectangles, only Stephen (A) responded favourably when he said that "*a square is a rectangle which has all sides to be equal. Yes, it is a special type of a rectangle*." This could mean that he had a good understanding of the inclusion relations between a square and a rectangle. In other words, Stephen understood that squares form a subset of rectangles. Findings about Stephen's justification support those of Rianasari et al. (2016) who also found their study participants using the phrase is a special form of to justify the inclusion of squares in rectangles. Cynthia incorrectly supported her claim from a quadrilateral point of view.

The remaining six participants said that a square was not a rectangle, and supported it with reasons like the following:

> *... for a square all the sides are equal and for rectangle opposite sides are equal* (Maxwell, A).
>
> *... the square is a type of a quadrilateral in which all the sides and the angles are the same but with that of a rectangle, their opposite sides are equal* (Albert, B).

Clement (A) smiled, sat quietly, and said, "*A square cannot be a rectangle. A rectangle has its opposite sides measuring equal but a square has all sides equal*."

> *… for square, all the sides are equal but, in a rectangle, only two of its sides are equal* (Alex, B).

The participants' responses show that their patterns of thinking were almost the same. They considered the shapes in isolation, not considering any possible relationship that could be drawn from the properties (Sfard, 2008). These responses suggest a ritual routine. Thus, these participants did not accept that a square could also be called a rectangle. Only one of the eight participants showed that a square was a rectangle. This finding supports that of Rianasari et al. (2016) who found that a few of the study participants, also PSTs, showed thinking that aligned with the concept of the inclusion relations of squares in rectangles.

Relationship Between a Square and a Rhombus

Only Clement (A) responded favourably that a square is a rhombus and justified as follows: "*a square is a rhombus. Their characteristics are almost the same. Four sides equal ..., both have opposite sides being parallel, diagonals bisecting at 90 degrees, a type of rhombus*."

For the remaining participants, it was difficult to accept that a square was also a rhombus.

Maxwell (A) justified his dismissive response by saying that:

> *for the square all the sides are equal. Even though for the rhombus, all the sides are equal, the angles in the square* [each] *is 90 degrees but for the rhombus the angles can be different. Opposite angles are the same.*

This shows Maxwell's and the other participants' ritual thinking. Maxwell thought that the fact that each angle in a square was 90° was different from the fact that opposite angles in a rhombus are the same.

Stephen and Jones (A) had performed quite well on the shape inclusion when compared with a parallelogram. However, data from the study shows that their understanding of the relationship among the other shapes was not well developed, or they had no knowledge about it. On the topic of the square and rhombus, Stephen said that "*a square is not a rhombus because in a square, all interior angles measure 90°, but in rhombus, we do not have that property*."

This response was similar to that of Maxwell (A). Jones (A) said that a square is not a rhombus, but he could not give any justification. Thus, the participants still considered the properties of shapes learned as isolated facts, with no effort to think about the relationships among them. This shows that their thinking about the inclusion of shapes is purely ritualistic (Sfard, 2008). Fujita and Jones (2006) report a similar finding that most PSTs indicated that a square was not a rhombus.

Relations Between a Rhombus and a Rectangle

The responses from the participants to the inclusion of a rhombus in a rectangle show that they have little or no knowledge of such a concept in geometry. The responses show their ritual thinking since they could not draw any relations concerning the sides of these figures, which give the necessary and sufficient conditions together with their properties (Rianasari et al., 2016). All the participants initially said that a rhombus was not a rectangle, even though Cynthia (B), after considering some properties, changed her decision with some level of uncertainty. She had been wrongly supporting her thinking of inclusion relations viewed from a quadrilateral as point of departure. During the interview, seeking to understand her thinking on the relationship between a rhombus and a rectangle, she said, "*I don't think so even though they are all four-sided figures*." She paused for a short while and said, "*Yes, it can be a rectangle because with rhombus we have two opposite angles equal like the rectangle*." Even though her response showed some level of uncertainty, it is possible that she may have begun to draw some connections between the properties of the two shapes.

The responses from those who said that a square was not a rhombus but used some related properties to justify their argument, are evidence of ritual thinking, since they could not make any logical deductions from the properties they had raised (Sfard, 2008). Almost all the participants justified their discourse by using the same properties, except for Alex (B), who said that he had no idea. For example, Jones (A) said, "*for a rectangle two opposite sides are equal in length but for a rhombus, all the sides are equal*." This probably shows that the properties of geometric figures are learned by rote with no understanding of any relationship among them, which is a demonstration of ritualised thinking within the framework (Sfard, 2008). What they might not have thought of was to make a logical deduction from the properties separate from the concept of parallelism. For example, consider the properties of the following quadrilaterals, A and B:

A: opposite sides are equal

B: all sides are equal

Question: Which property is a subset of the other?

Answering the above question demands critical thinking. It may be conjectured that it is for these critical thinking, reasoning, and problem-solving abilities that the teaching and learning of geometry is being given more attention in the mathematics curriculum (Jones, 2000; NCTM, 2000). Possibly, if the participants had engaged in such a logical deduction from the properties, their thinking about inclusion relations may have improved.

Conclusion

This study was guided by two research questions with which we sought to determine the participating PSTs' ability to define quadrilaterals and show an understanding of inclusion relations among them. The results show that the PSTs generally employed limited geometric thinking when defining some types of quadrilaterals. Notable among them were the rhombus, trapezium and kite. It is expected that PSTs should be able to define geometric shapes to develop competencies in teaching these types of quadrilaterals since definitions play a significant role in geometric education and mathematics as a whole (Luneta, 2015). Learners will need to know what an object is in order to understand its properties. The results also show that the PSTs had difficulty understanding and classifying quadrilaterals. For example, many of them found it difficult to accept that a square was a rectangle or a parallelogram. Many of the PSTs found it difficult to exercise their geometric thinking about the relationships between the geometric properties of the quadrilaterals, and which of them shared certain properties. For example, almost all the participants' acceptance that a rhombus was a parallelogram was based on the physical appearance of these two shapes and not on their properties. The PSTs in Group A (even though only a few) showed advanced competencies in the definition and classification of quadrilaterals compared to their counterparts in

Group B, many of which demonstrated limited understanding thereof. We recommend that mathematics teachers need to develop pre-service mathematics teachers' understanding of geometric definition and classification of quadrilaterals to serve as foundational knowledge for the teaching of advanced related topics. This will help improve their geometric competencies about the definition and classification of quadrilaterals to enhance their classroom delivery of these concepts.

Authors' Contributions

EL wrote the manuscript and VM reviewed the final manuscript.

Notes

ii. DATES: Received: 30 November 2022; Revised: 19 March 2025; Accepted: 25 July 2025; Published: 31 December 2025.

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